Root Locus Diagram

Objective

Upon completion of this chapter you will be able to:

- Plot the Root Locus for a given Transfer Function by varying gain of the system,
- Analyse the stability of the system from the root locus plot.
- Determine all parameters related to Root Locus Plot.
- Plot Complimentary Root Locus for negative values of Gain.
- Plot Root Contours by varying multiple parameters.

Introduction

The transient response of a closed loop system is dependent upon the location of closed loop poles. If the system has a variable gain then location of closed loop poles is dependent on the value of gain. It is therefore necessary that we must know the how the location of closed loop varies with change in the value of loop gain is varied. In control system design it may help to adjust the gain to move the closed loop poles at desired location. So, root locus plot gives the location of closed loop poles as system gain $K$ is varied.

Root loci: The portion of root locus when $k$ assume positive values: that is $0 \leq k < \infty$

Complementary Root loci: The portion of root loci $k$ assumes negative value, that is $-\infty < k \leq 0$

Root contours: Root loci of when more than one system parameter varies.

Criterion for Root Loci

Angle condition: It is used for checking whether any point is lying on root locus or not & also the validity of root locus shape for closed loop poles.

For negative feedback systems, the closed loop poles are roots of Characteristic Equation

$$1 + G(s)H(s) = 0$$

$$G(s)H(s) = -1$$

$$\angle \left( G(s)H(s) \right) = 180^0 - \tan^{-1}(0) = 180^0 = \pm 180^0 = (2q + 1)180^0$$
Angle condition may be started as, for a point to lie on root locus, the angle evaluated at that point must be an odd multiple of ±180°.

**Magnitude condition:** It is used for finding the magnitude of system gain k at any point on the root locus.

\[
1 + G(s)H(s) = 0
\]

\[
G(s)H(s) = -1
\]

\[
\left| G(s)H(s) \right| = 1
\]

**Solved Examples**

**Problem:** Determine whether the points \( s_1 = -3 + 4j \) & \( s_2 = -3 - 2j \) lie on the root locus of

\[
G(s)H(s) = \frac{k}{(s + 1)^4}
\]

**Solution:**

\[
G(s)H(s) \bigg|_{s = s_1} = \frac{k}{[ -3 + 4j + 1 ]^4} = \frac{k}{[ -2 + 4j ]^4}
\]

\[
\angle (G(s)H(s)) = 0° - 4 \times \left( 180° - 63.43° \right) = -464° = \text{Not an odd multiple of } 180°
\]

\[
G(s)H(s) \bigg|_{s = s_2} = \frac{k}{[ -3 - 2j + 1 ]^4} = \frac{k}{[ -2 - 2j ]^4}
\]

\[
\angle (G(s)H(s)) = 0° + 135° \times 4 = 540° = \left( 3 \times 180° \right) = \text{Odd multiple of } 180°.
\]
Advantages of Root Locus

- The root locus is a powerful technique as it brings into focus the complete dynamic response of the system and further, being a graphical technique, an approximate root locus can be made quickly and designer can be easily visualize the effect of varies system parameters an root location.

- The root locus also provides a measure of sensitively of roots to the vacations in parameters being considered.

- It may further be pointed out here that root locus technique is applicable to single as well as multiple loop system.

Construction Rules Of Root Locus

**Rule 1:** The root locus is symmetrical about real axis \((G(s)H(s) = -1)\)

**Rule 2:** Each branch of Root Locus originates at an open loop pole and terminates at an open loop zero or infinity.

Let \(P\) = number of open loop poles

\(Z\) = number of open loop zeroes

And if \(P > Z\)

Then,

No. of branches of root locus = \(P\)

No. of branches terminating at zeroes = \(Z\)

No. of branches terminating at infinity = \(P - Z\)

**Rule 3:** A point on real axis is said to be on root locus if to the right side of their point the sum of number of open loop poles & zeroes is odd.

\[G(s) = \frac{k(s+2)(s+6)}{s(s+4)(s+8)}\]

\(P = 3\); \(Z = 2\); \(P - Z = 1\)

NRL – Not Root Locus; RL – Root Locus
Rule 4: Angle of asymptotes

The P – Z branches will terminate at infinity along certain straight lines known as asymptotes of root locus.

Therefore, number of asymptotes = (P - Z)

Angle of asymptotes is given by: \( \theta = \frac{[2q + 1]180^\circ}{P - Z} \); q = 0,1,2,3……

Suppose if P – Z = 2, then angle of asymptotes would be:

\[
\theta_1 = \frac{2(0) + 1}{2} \times 180^\circ = 90^\circ \quad \text{and} \quad \theta_2 = \frac{2(1) + 1}{2} \times 180^\circ = 270^\circ
\]

Solved Examples

Problem: Find the angle of asymptotes of a system whose characteristic equation is 
\( s(s + 4)(s^2 + 2s + 1) + k(s + 1) = 0 \)

Solution: The characteristic equation can be written in standard form as shown below:

\[
1 + \frac{K(s + 1)}{s(s + 4)(s^2 + 2s + 1)} = 0
\]

\[
G(s)H(s) = \frac{K(s + 1)}{s(s + 4)(s^2 + 2s + 1)}
\]

P = 4 ; Z = 1 ; P – Z = 3

\[
\theta_1 = \frac{2(0) + 1}{3} \times 180^\circ = 60^\circ
\]

\[
\theta_2 = \frac{2(1) + 1}{3} \times 180^\circ = 180^\circ
\]

\[
\theta_3 = \frac{2(2) + 1}{3} \times 180^\circ = 300^\circ
\]
Rule 5: Centroid

It is the intersection point of asymptotes on the real axis, it may or may not be a part of root locus.

\[
\text{centroid} = \frac{\sum (\text{Real part of open loop poles}) - \sum (\text{Real part of open loop zeroes})}{(P - Z)}
\]

Solved Examples

Problem: Find centroid of the system whose characteristic equation is given as:

\[s^3 + 5s^2 + (K + 6)s + k = 0\]

Solution: \[s^3 + 5s^2 + sk + k + 6s = 0 \Rightarrow s^3 + 5s^2 + 6s + k(s + 1) = 0\]

\[
1 + \frac{k(s + 1)}{s^3 + 5s^2 + 6s} = \frac{k(s + 1)}{s(s + 2)(s + 3)}
\]

\[P = 3, Z = 2\]

Poles = 0, -2, -3 ; zeroes = -1

\[
\text{centroid} = \frac{(0-2-3)-(-1)}{3-1} = [-2, 0]
\]

Rule 6: Break Points

They are those points where multiple roots of the characteristics equation occur.

Procedure to find Break Points:

1. Construct \(1 + G(s)H(s) = 0\)
2. Write \(k\) in terms of \(s\)
3. Find \(\frac{dk}{ds} = 0\)
4. The roots of \(\frac{dk}{ds} = 0\) give the locations of break-away points.
5. To test the validity of breakaway points substitute in step – 2.

If \(k > 0\), it means a valid breakaway point.
**Breakaway Points**

If the break point lies between two successive poles then it is termed as a Breakaway Point.

**Predictions**

1) The branches of root locus either approach or leave the breakaway points at an angle of
\[ \pm \frac{180}{n} \]

Where \( n \) = no. of branches approaching or leaving breakaway point.

2) The complex conjugate path for the branches of root locus approaching or leaving or breakaway points is a circle.

3) Whenever there are 2 adjacently placed poles on the real axis with the section of real axis between there as part of RL, then there exists a breakaway point between the adjacently placed poles.

**Solved Examples**

**Problem:** Plot Root Locus for the system shown below:

**Solution:** Applying Construction Rules of the Root Loci

\[
\frac{C(s)}{R(s)} = \frac{k}{s^2 + 2s + k}
\]

\[
G(s) = \frac{k}{s(s+2)}
\]

**Rule 2:** \( P = 2, Z = 0, P - Z = 2 \)

**Rule 3:** Section of real axis lying on Root Locus
Rule 4: \( \theta_1 = \frac{(2(0)+1)}{2} \times 180 = 90^0; \ \theta_2 = \frac{(2(1)+1)}{2} \times 180 = 270^0 \)

Rule 5: centroid = \( \frac{0 + (-2) - 0}{2} = -1 \)

Rule 6: Breakaway points

\[ s^2 + 2s + k = 0 \implies k = \left( -s^2 - 2s \right) \]

\[ \frac{dk}{ds} = -2s - 2 = 0 \implies s = -1 \]

From the third prediction about Breakaway points we know that there must be a breakaway point between \( s=0 \) and \( s=-2 \). Hence \( s=-1 \) is a valid breakaway point.

Since centroid and breakaway point coincide the branches of root locus will leave breakaway point along the asymptotes.

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**Break – in points**

The points where two branches of roots locus converge are called as break in points. A Break-in Point lies between two successive zeroes.

To differentiate between break in & break away points.

\[ \frac{d^2k}{ds^2} > 0 \] For break in points

\[ \frac{d^2k}{ds^2} < 0 \] For break-away points
Otherwise by observation a breakaway point lies between two poles and a break-in point lies between two zeroes.

**Predictions about Break-in Points:**

1. Whenever there are 2 zeroes on real axis & the portion of real axis between 2 zeroes lies on root locus then there is a break in points between 2 zeroes.

![Break-in Point Diagram](image)

2. Whenever there exists a zero on real axis & real axis on left is on root locus & P > Z, then there will be a break in to left of zero.

![Effect of Adding Poles](image)

**Effect of adding poles to a transfer function**

Suppose we add a pole at s=-4 to our previous Transfer Function $G(s) = \frac{K}{s(s+2)}$

The new transfer function will be:

$G(s) = \frac{K}{s(s+2)(s+4)}$

**Rule 2:** P = 3, Z = 0, P - Z = 3
Rule 3: The sections of real axis lying on Root Locus is shown on right

Rule 4: \( \theta_1 = 60^0, \ \theta_2 = 180^0, \ \theta_3 = 300^0 \)

Rule 5: Centroid

\[ \sigma = \frac{[0 + (-2) + (-4) - 0]}{3 - 0} = -2 \]

Rule 6: Break away points

Characteristic Equation: \( s^3 + 6s^2 + 8s + K = 0 \) \implies \( K = -s^3 - 6s^2 - 8s \)

\[ \frac{dK}{ds} = -3s^2 - 12s - 8 = 0 \]

\( 3s^2 + 12s + 8 = 0 \implies s = -0.845, -3.15 \)

Since Breakaway point must lie between two successive poles that is it must lie between 0 and -2 so \( s = -0.845 \) is a valid breakaway point whereas \( s = -3.15 \) is invalid.

The root locus for this system is shown below,

Here, the root locus shifts towards right & hence stability decreases.
Effect of adding zeroes to a transfer function

Suppose we add a zero to the transfer function in the previous case at \( s = -1 \), then the Transfer Function becomes:

\[
G(s) = \frac{K(s + 1)}{s(s + 2)(s + 4)}
\]

Rule 2: \( P = 3, \ Z = 1, \ P - Z = 2 \)

Rule 3: The sections of real axis lying on Root Locus are shown below:

Rule 4: \( \theta_1 = 90^0, \ \theta_2 = 270^0 \)

Rule 5: Centroid = \[
\frac{0 + (-2) + (-4) - (-1)}{2} = -2.5
\]

Rule 6: Breakaway point

Characteristic Equation of Closed Loop System is: \( s^3 + 6s^2 + 8s + Ks + K = 0 \) =>

\[
K = \frac{-s^3 - 6s^2 - 8s}{s + 1}
\]

\[
\frac{dk}{ds} = \frac{(s + 1)(-3s^2 - 12s - 8) + (s^3 + 6s^2 + 8s)}{(s + 1)^2} \bigg|_{0} = s = -2.9
\]
Control Systems (Root Locus Diagram)

The branches originate at Breakaway Point and tend to infinity along the asymptotes. Here, the root locus shifts towards left and hence stability increase.

**Solved Examples**

**Problem:** Draw Root Locus for the system whose open loop transfer function is

\[ G(s) = \frac{K(s + 2)}{s(s + 1)} \]

**Solution:** Applying Construction rules of the Root Loci

**Rule 2:** \( P = 2, Z = 1, P - Z = 1 \)

**Rule 3:** The sections of real axis lying on Root Locus are shown below:

**Rule 4:** Angle of Asymptotes \( \theta = 180^0 \)

**Rule 5:** centroid = \( \frac{0 + (-1) - (-2)}{1} = 1 \)

**Rule 6:** Break away points

\[ s(s + 1) + K(s + 2) = 0 \Rightarrow K = \frac{-s^2 - s}{s + 2} \]

\[ \frac{dk}{ds} = 0 = \frac{(s + 2)(-2s - 1) + (s^2 + s)(1)}{(s + 2)^2} \]

\[ s^2 + 4s + 2 = 0 \Rightarrow s = (-2 \pm \sqrt{2}) = -0.6, -3.4 \]

Since, \( s = -0.6 \) lie between two consecutive poles it is a Break away point and since \( s = -3.4 \) lies between a zero and infinity it is a break-in point.
So, the root locus looks like as shown below:

**Proof of path being a circle**

\[
\frac{k(s + b)}{s(s + a)} \quad \text{Let } s = (x + jy)
\]

\[
\frac{K[x + jy + b]}{[x + jy][x + jy + a]} = \frac{K[(x + b) + jy]}{x^2 + 2jxy + ax + jay - y^2} = \frac{K[(x + b) + jy]}{x^2 + ax - y^2 + j[2xy + ay]}
\]

For \( s = (x + jy) \) lies on the root locus, angle should be odd multiple of 180°

\[
\tan^{-1}\left(\frac{y}{x + b}\right) - \tan^{-1}\left(\frac{2xy + ay}{x^2 + ax - y^2}\right) = 180^0
\]

Taking ‘tan’ on both sides

\[
\frac{y}{x + b} - \frac{2xy + ay}{x^2 + ax - y^2} = 0
\]

\[
x^2 + ax - y^2 - (2x + a)(x + b) = 0 \implies -x^2 - 2bx - y^2 - ab = 0
\]

\[
(x + b)^2 + y^2 = b(b - a) \quad \text{centre} = (-b, 0)
\]

\[
\text{radius} = \sqrt{b(b - a)}
\]

So, in such a case the centre of circular trajectory of Root Locus lies at the zero of Open Loop System.
Rule 7: Intersection of Root Locus with imaginary axis

Roots of auxiliary equation $A(s)$ at $k = k_{\text{max}}$ (i.e. Maximum value of Gain $K$ for which the closed loop system is Stable) from Routh Array gives the intersection of Root locus with imaginary axis.

The value of $k = k_{\text{max}}$ is obtained by equating the coefficient of $s^1$ in the Routh Array to zero.

Solved Examples

Problem: Find the intersection of root locus with the imaginary axis and also the intersection of asymptotes with imaginary axis for the system whose open loop Transfer Function is given by $G(s) = \frac{k}{s(s+2)(s+4)}$.

Solution: Characteristic Equation of the system is given as

$$s^3 + 6s^2 + 8s + k = 0$$

Routh Array:

| $s^3$ | 1   | 8   |
| $s^2$ | 6   | $k$ |
| $s^1$ | $48 - k$ | 6   |
| $s^0$ | $k$   |

For stability $\frac{48 - k}{6} > 0 = k < 48 \& k > 0$

$\therefore$ range $0 < k < 48$

At $k = k_{\text{max}} = 48$

$A(s) = 6s^2 + k = 0 \Rightarrow 6s^2 + 48 = 0$

$s = \pm j\sqrt{8} = \pm j2.82$
Shortcut method

If the Transfer Function is of the type

\[ G(s) = \frac{K}{s(s+a)(s+b)} \]

Intersection of RL with jω axis is given by \( s = \pm j\sqrt{ab} \)

Intersection of asymptotes with jω axis

Since we are aware about the angle of asymptote from Rule-4 and also the x-axis intercept of the asymptote which is centroid so we can find y-intercept as shown below:

\[ \tan \theta = \frac{y}{x} \]
\[ \tan 60^\circ = \frac{y}{2} \]
\[ y = 2\sqrt{3} = 3.4 \]

To find Gain, \( k \) at any point on root locus geometrically

\[ k = \frac{\text{Product of vector length of poles}}{\text{Product of vector length of zeroes}} \]

Suppose, we need to find Gain \( k \) of the root locus plot for \( \zeta = 0.5 \), then we need to find the roots of characteristic equation corresponding to \( \zeta = 0.5 \), which can be done by finding intersection of root locus with a straight line oriented \( \theta \) from –ve x axis in clockwise direction and passing through origin.

\[ \theta = \cos^{-1} \zeta = 60^\circ \]

Gain, \( k = \frac{\text{Product of vector length of poles}}{\text{Product of vector length of zeroes}} \frac{p_1 \times p_2 \times p_3}{1} \]
Rule 8: Angle of Departure and Arrival

The angle made by root locus with real axis when it departs from a complex open loop poles is called as angle of departure.

The angle made by root locus with real axis when it arrives at complex open loop zero is called as angle of arrival.

\[ \phi_D = 180 + \angle GH' \]

\[ \phi_A = 180 - \angle GH' \]

\( \angle GH' \) = angle of the function excluding the concerned poles at the poles itself.

Solved Examples

Problem: Plot Root Locus for the system whose open loop transfer function is given by:

\[ G(s)H(s) = \frac{k}{s(s + 4)(s^2 + 4s + 20)} \]

Solution: Rule 2: \( P = 4, Z = 0, P - Z = 4 \)

Rule 3: The section of real axis lying on root locus is shown in the adjoining figure.
Rule 4: \( \theta_1 = 45^0, \theta_2 = 135^0, \theta_3 = 225^0, \theta_4 = 315^0 \)

Rule 5: centroid = \( \frac{0 + (-2) + (-2) + (-4) - 0}{4} = -2 \)

Rule 6: Breakaway points

\[
s^4 + 8s^3 + 36s^2 + 80s + K = 0
\]

\[
K = -s^4 - 8s^3 - 36s^2 - 80
\]

\[
\frac{dk}{ds} = 0 = 4s^3 + 24s^2 + 72s + 80 = 0 \Rightarrow s = -2, -2 \pm j2.45
\]

Note: To check validity of complex break point use angle criterion.

Rule 7: Routh Array

\[
\begin{array}{cccc}
 s^4 & 1 & 36 & K \\
 s^3 & 8 & 80 & 0 \\
 s^2 & 26 & K \\
 s^1 & \frac{2080 - 8k}{26} \\
 s^0 & K \\
\end{array}
\]

For stability \( \frac{2080 - 8k}{26} > 0 \Rightarrow k < 260 \& K > 0 \)

\[
\therefore \text{range } 0 < k < 260
\]

For \( k = 260 \) A.E. = \( A(s) = 26s^2 + k = 0 \)

\[
26s^2 + 260 = 0 \Rightarrow s = \pm j3.16
\]

Intersection of asymptotes with \( j\omega \) – axis

\[
y = \tan 45 \times 2 = \pm j2
\]
Rule 8: Angle of departure

\[ \phi_{p1} = 180 - \tan^{-1} \left[ \frac{4 - 0}{0 - (-2)} \right] = 180 - \tan^{-1} 2 = 116.56^0 \]

\[ \phi_{p2} = 90^0 \]

\[ \phi_{p3} = \tan^{-1} \left[ \frac{4 - 0}{(-2)(-4)} \right] = 63.4^0 \]

\[ \phi = \Sigma \phi_z - \Sigma \phi_p \]

\[ \phi_D = 180 + \phi = 180 - [116.6 + 90 + 63.4] = 180 - 270 = -90^0 \]

Else

\[ G(s)H(s) \bigg|_{s = -2 + 4j} = \frac{K}{s(s + 4)(s + 2 + 4j)(s + 2 - 4j)} = \frac{K}{(-2 + 4j)(2 + 4j)(8j)} \]

\[ \angle GH' = -\tan^{-1} \left( \frac{4}{-2} \right) - \tan^{-1} \left( \frac{4}{2} \right) - 90^0 = -\left( 180 - \tan^{-1} \left( \frac{4}{2} \right) \right) - \tan^{-1} \left( \frac{4}{2} \right) - 90^0 = -270 \]

\[ \phi_D = 180 - 270 = -90^0 \]

So, the root locus of the system looks like as shown below:
Problem: Plot Root Locus for the system whose Open Loop Transfer Function is given by

\[ G(s) = \frac{k(s^2 - 2s + 5)}{(s + 2)(s - 0.5)} \]

Solution: Rule 2: \( P = 2, Z = 2, P - Z = 0 \)

Rule 3: The section of real axis lying on Root Locus is shown below:

Since \( P - Z = 0 \), Rule 4 and Rule 5 are of no use as there are no asymptotes.

Rule 6: Breakaway points

Characteristic Equation of the system is:

\[(s + 2)(s - 0.5) + k(s^2 - 2s + 5) = 0 \Rightarrow k = \frac{-(s^2 + 1.5s - 1)}{(s^2 - 2s + 5)}\]

\[\frac{dk}{ds} = 0 \Rightarrow s = -0.4, 3.6\]

Here, since Breakaway point must lie between two consecutive poles so \( s = -0.4 \) is a valid Breakaway Point whereas \( s = 3.6 \) is an invalid point.

Rule 7: \( s^2 (1 + k) + s(1.5 - 2k) + (5k - 1) = 0 \)

Routh Array:

<table>
<thead>
<tr>
<th>s^2</th>
<th>1 + k</th>
<th>5k - 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^1</td>
<td>1.5 - 2k</td>
<td>0</td>
</tr>
<tr>
<td>s^0</td>
<td>5k - 1</td>
<td>0</td>
</tr>
</tbody>
</table>

For stability \( 1 + k > 0 \Rightarrow k > -1 \) & \( 1.5 - 2k > 0 \Rightarrow k < 0.75 \) & \( 5k - 1 > 0 \Rightarrow k > 0.2 \)

\[ \therefore \text{range } 0.2 < k < 0.75 \]
Control Systems (Root Locus Diagram)

\[ k = k_{\text{max}} = 0.75 \]

A.E. = \( A(s) = (1+k)s^2 + (5k-1) = 1.75s^2 + (5 \times 0.75 - 1) = 0 \)

\[ s = \pm j1.25 \]

**Rule 8: Angle of arrival**

\[ G(s) = \frac{k(s-1+2j)(s-1-2j)}{(s+2)(s-0.5)} \]

\[ \phi_A \text{ at } s = (1+2j) \]

\[ \angle GH' = \angle \left( \frac{1+2j-1+2j}{3+2j} \right) (0.5+2j) = \tan^{-1} (4j) - \tan^{-1} \left( \frac{2}{3} \right) - \tan^{-1} \left( \frac{2}{0.5} \right) \]

\[ \angle GH' = 90^0 - \tan^{-1} \left( \frac{2}{3} \right) - \tan^{-1} \left( \frac{2}{0.5} \right) = -20^0 \]

\[ \phi = 180^0 + 20^0 = 200^0 \]

The Root Locus for the system looks like as shown below:
Analysis of system having dead time or transportation lag

Dead Time: It is one of the form of non-linearity and is defined as the time in which a system does not respond to change in input. It is approximated as a zero in RHS s-plane.

Transfer function having poles or zeroes in RHS s-plane are called as non-minimum phase functions.

For curve – 1
Output \( y(t) = \text{input } x(t) \)

For curve – 2
\( y(t) = x(t - T) \)

Applying Laplace Transform both sides
\[
Y(s) = e^{-sT} \times s(s + 3)
\]

Time domain approximation
\[
y(t) = x(t - T) = x(t) - T \dot{x}(t) + \frac{T^2}{2!} \ddot{x}(t) - \cdots \]

\[
y(t) = x(t) - T \dot{x}(t)
\]

\[
Y(s) = X(s) - sTX(s) = X(s)(1 - sT) = X(s)e^{-sT}
\]

\[
e^{-sT} = (1 - sT)
\]

Ex: \( G(s) = \frac{ke^{-s}}{s(s+3)} = \frac{k(1-s)}{s(s+3)} \)
Complimentary Root Locus (CRL)

Angle Condition:
\[ \angle G(s)H(s) = 0^0 = \pm [2q]180^0 \]

Magnitude Condition:
\[ |G(s)H(s)| = 1 \]

Construction Rules

Rule 1: The CRL is symmetrical about real axis \( G(s)H(s) = 1 \)

Rule 2: Number of branches terminating at infinity is \((P - Z)\).

Rule 3: A point on real axis is said to be on CRL if to the right side of the point the sum of open loop poles & zeroes is even.

Rule 4: Angle of asymptotes
\[ \theta = \frac{[2q]180^0}{(P - Z)}, \quad q = 1, 2, 3, \ldots \]

Rule 5: Centroid: same as Root locus

Rule 6: Break point: Same as RL

Rule 7: Intersection of CRL with \( j\omega \) axis same as root locus.

Rule 8: Angle of departure & arrival.
\[ \phi_D = 0 + \phi \]
\[ \phi_A = 0 - \phi \]

Where \( \phi = \sum \phi_Z - \sum \phi_P = \angle GH' \)

Solved Examples

Problem: Plot Root Locus for the system whose Open Loop Transfer Function is
\[ G(s) = \frac{Ke^{-s}}{s(s + 3)} \]
Solution:

\[ G(s) = \frac{k(1-s)}{s(s+3)} = \frac{-k(s-1)}{s(s+3)} \]

So, here the gain becomes negative and we thus plot the Complimentary Root Locus.

**Rule 2:** \( P = 2, Z = 1, \ P - Z = 1 \)

**Rule 3:** The section of real axis lying on Root Locus is shown in the figure below:

![Root Locus Diagram](image)

**Rule 6: Breakaway points**

Characteristic Equation of the system is:

\[ 1 + \frac{K(1-s)}{s(s+3)} = 0 \]

\[ s(s+3) + k(1-s) = 0 \implies k = \frac{-s^2 - 3s}{1-s} \]

\[ \frac{dk}{ds} = 0 = \frac{(1-s)(-2s-3) - [(-s^2 - 3s)(-1)]}{(1-s)^2} \]

\[ s^2 - 2s - 3 = 0 \implies s = 3, -1 \]

Since Breakaway point must lie between two consecutive poles, so it must lie between 0 and -3 and hence \( s = -1 \) is a valid breakaway point.

**Rule 7: Intersection of Root Locus with Imaginary Axis**

Characteristic Equation: \( s^2 + s(3-k) + k = 0 \)
Routh Array:
\[
\begin{array}{ccc}
 s^2 & 1 & k \\
 s^1 & 3 - k & 0 \\
 s^0 & k & \\
\end{array}
\]

For stability $3-k > 0 \Rightarrow k < 3 \& k > 0$

Range $0 < k < 3$

$k = k_{\text{max}} = 3$

A.E. = $A(s) = s^2 + k = 0$

$s = \pm j1.732$

**Root Contours**

Root contours are multiple root locus diagrams obtained by varying multiple parameters in a transfer function diagram in same $s$ – plane.

**Case 1:** let $\alpha = 0$, and gain $K$ is varied

$G(s)H(s) = \frac{k(s + \alpha)}{s(s + 1)(s + 8)}$

$G(s) = \frac{k}{(s + 1)(s + 8)}$
**Case 2:** In this case, gain $K=1$ and $\alpha$ is varied

For plotting Root Locus with respect to $\alpha$, we must first manipulate Characteristic Equation in such a way that $\alpha$ acts as gain of the system.

\[
1 + \frac{k(s + \alpha)}{s(s + 1)(s + 8)} = 0 \Rightarrow s(s + 1)(s + 8) + k(s + \alpha) = 0
\]

\[
1 + \frac{k\alpha}{s(s + 1)(s + 8) + ks} = 0
\]

\[
1 + G(s)H(s) = 0
\]

\[
G(s)H(s) = \frac{k\alpha}{s(s + 1)(s + 8) + ks}
\]

Put $k = 1$

\[
G(s)H(s) = \frac{\alpha}{s(s^2 + 9s + 9)}
\]